Trace-distance measure of Coherence

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Joint works...

Collaborators: Maciej Lewenstein, Swapan Rana, and Andreas Winter

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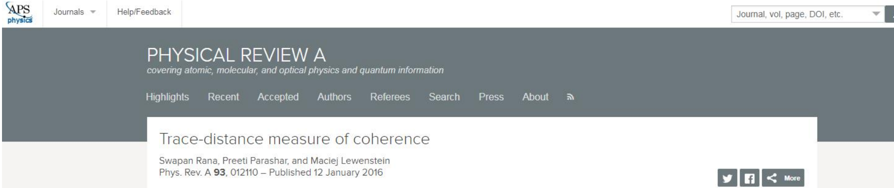

Outline

- **Introduction to coherence measures**
  - Model: Incoherent Operations (IO)
  - Coherence measures: basic requirements
- **Distance-based measures**
  - ℓ_p - and Shatten-p-norm based measures?
  - Neither is strong monotone for $p > 1$
- **Trace distance measure of coherence**
  - Strong monotone for Qubits and X-states
  - But not for all states
  - What about entanglement?
- **Operational interpretations of C_{ℓ_1}**
  - Equals to Negativity of the MC state
  - Optimum C_r for a given C_{ℓ_1}?
- **Conclusion**

Coherence Theory

Resource Theory of Coherence [Baumgratz *et al.*, PRL (2014)]

- **Free states: Diagonal in chosen basis**
  - 'Preferred', or 'Chosen' basis $\{|i\rangle\}$ of the corresponding Hilbert space $\mathcal{H} = \mathbb{C}^d$.
  - Set of 'incoherent', or 'free' states:

$$\mathcal{I} = \left\{ \delta \,\middle|\, \delta = \sum_i p_i |i\rangle\langle i| \right\}.$$

- **Free Operations: Incoherent Operations (IO)**
  - An operator K is 'incoherent' if $K\delta K^\dagger \in \mathcal{I}$, $\forall \delta \in \mathcal{I}$.
    $\implies K$ can have at most one non-zero entry in any column.
  - A CPTP map Λ is 'free' or 'incoherent' if $\exists$ incoherent Kraus operators $\{K_n\}$ for Λ.

Coherence measures

What makes a coherence measure?

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Coherence Monotones

Any function $C\colon \mathscr{B}(\mathscr{H}^d) \to \mathbb{R}^+$ is a coherence measure if it satisfies

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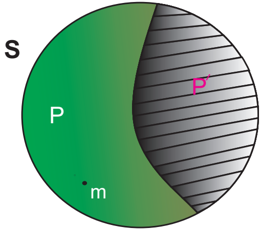
3a. Monotonicity: $C(\Lambda_{\text{ICPTP}}[\rho]) \leq C(\rho)$, $\forall \rho$ and incoherent Λ.

3b. Strong Monotonicity: $\sum p_n C(\rho_n) \leq C(\rho)$, $\rho_n = K_n \rho K_n^\dagger / p_n$, $p_n = \text{Tr}[K_n \rho K_n^\dagger]$, for all incoherent $\{K_n\}$. Allows selective measurements or stochastic operations. Harder to verify: $3\text{b}. + 2. \Rightarrow 3a.$

Distance-based measures

How to quantify (measure)?

Let $m \in S$ and P be a *property* regarding the members of S. The usual way to quantify how much of the property P is contained in m is to determine the distance of m from the set of all $x \in S$ which does not have the property P.



Entanglement from contractive metric [Vedral and Plenio, PRA 1998]

If a *distance function* $\mathscr{D}$ satisfies

i. Positivity: $\mathscr{D}(\rho,\sigma) \geq 0 \quad \forall\ \rho,\sigma$, with equality iff $\rho = \sigma$,

ii. Contractivity: $\mathscr{D}(\Lambda[\rho],\Lambda[\sigma]) \leq \mathscr{D}(\rho,\sigma)$ for all CPTP map Λ,

then an entanglement measure can be defined through this distance as

$$E(\rho) = \inf_{\sigma \in \{\text{Separable states}\}} \mathscr{D}(\rho,\sigma).$$

- ✓ REE :

$$\mathscr{D}(x,y) = S(x\|y) := \begin{cases} \mathrm{Tr}(x\log x - x\log y), & \text{if support } x \subseteq \text{ support } y \\ +\infty, & \text{otherwise} \end{cases}$$

- ✓ Bures metric: $\mathscr{D}(x,y) = 2 - 2\sqrt{F(x,y)}$, where $F(x,y) := [\mathrm{Tr}\{\sqrt{x}y\sqrt{x}\}^{1/2}]^2$
- ✓ Trace distance: $\mathscr{D}(x,y) = \|x-y\|_1 := \mathrm{Tr}\,|x-y|$

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- ✓ Trace distance: $\mathscr{D}(x,y)=\|x-y\|_1:=\mathrm{Tr}|x-y|$
- ✖ Hilbert-Schmidt distance: $\mathscr{D}(x,y)=\|x-y\|_2$ is *not-contractive* [Ozawa, PLA 2000].

Measures from ℓ_p- and Schatten-p-norm (C_{ℓ_p}, C_p)

- For $X \in \mathbb{C}^{m\times n}$, $p \in [1,\infty)$, the norms are defined by

$$\|X\|_{\ell_p} := \left(\sum_{i,j} |x_{ij}|^p\right)^{1/p},$$

$$\|X\|_p := \left(\mathrm{Tr}|X|^p\right)^{1/p} = \left(\sum_i^r \sigma_i^p\right)^{1/p}, \sigma = \lambda\left(\sqrt{X^\dagger X}\right).$$

- The induced distance functions C_{ℓ_p} and C_p satisfies 1. and 2.
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For all $p \in (1, \infty)$ neither C_{ℓ_p} nor C_p is a monotone [PRA (2016)].

None of C_{ℓ_p}, C_p is a monotone for $p > 1$

Counterexample from any $\rho \notin \mathcal{I}$

Let $K_i = |0\rangle\langle i-1|$, $i = 1, 2, \ldots, d$ and consider the incoherent channel $\Lambda = \{\tilde{K}_i\}$ with the Kraus operators $\tilde{K}_i = \mathbb{1} \otimes K_i$. Then we have

$$\begin{aligned} C_p\left(\Lambda[\rho \otimes \mathbb{1}/d]\right) &= C_p\left(\rho \otimes |0\rangle\langle 0|\right) \\ &= C_p\left(\rho\right) \\ &> C_p\left(\rho \otimes \mathbb{1}/d\right). \end{aligned}$$

The inequality follows from

$$C_p\left(\rho \otimes \mathbb{1}/d\right) \le \|\rho \otimes \mathbb{1}/d - \delta^\star \otimes \mathbb{1}/d\|_p = C_p(\rho)\|\mathbb{1}/d\|_p < C_p(\rho).$$

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As $C_{\ell_p}(\rho \otimes \mathbb{1}/d) = d^{1/p-1} C_{\ell_p}(\rho) < C_{\ell_p}(\rho)$, C_{ℓ_p} also violates monotonicity.

Hence C_{ℓ_p}, C_p is not strong monotone for $p > 1$

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$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \\ \bar{a} & 0 & 1 & 0 \\ 0 & \bar{b} & 0 & 1 \end{pmatrix}, K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\rho_1 = \frac{1}{4p_1}\begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix} \oplus 0,\ \rho_2 = \frac{1}{4p_2}\begin{pmatrix} 1 & b \\ \bar{b} & 1 \end{pmatrix} \oplus 0,\ p_1 = p_2 = \frac{1}{2}.$$

The strong monotonicity (for both C_{ℓ_p} and C_p): $(|a| + |b|)^p \leq |a|^p + |b|^p$.

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$$(x+y)^{1+\epsilon}=(x+y)(x+y)^{\epsilon}=x(x+y)^{\epsilon}+y(x+y)^{\epsilon}>x\cdot x^{\epsilon}+y\cdot y^{\epsilon}=x^{1+\epsilon}+y^{1+\epsilon}$$

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Strong monotonicity does not hold for all $ab \neq 0$ and $p \in (1, \infty)$.

The only remaining case $p=1$: C_{ℓ_1} and C_{Tr}

$$C_{\ell_1}(\rho) := \min_{\delta\in\mathcal{I}} \|\rho-\delta\|_{\ell_1} = \sum_{i\neq j} |\rho_{ij}|$$

$$C_{\text{Tr}}(\rho) := \min_{\delta\in\mathcal{I}} \|\rho-\delta\|_1 = ?$$

- C_{ℓ_1} is a (strong) monotone [Baumgratz *et al.*, PRL (2014)].
- C_{Tr} is a monotone, as $\|.\|_1$ is contractive under any CPTP map.

Semidefinite Program (SDP) for $C_{\text{Tr}}(\rho)$ [PRA (2016)]

Minimize $\text{Tr}(P+N)$

subject to
$$\begin{cases} P-N &= \rho-\delta, \\ \text{Tr}\,\delta &= 1, \\ \delta &\in \mathcal{I} \\ P, N, \delta &\geq 0. \end{cases}$$

For pure states, a better SDP: [Chen *et al.*, PRA (2016)].

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Strong coherence monotone: any convex and monotone C is strong monotone iff $C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$.

For $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum |i\rangle$, $C_{\mathrm{Tr}}(|\Phi_d\rangle) = 2(1 - 1/d)$. Choosing $\rho_1 = |\Phi_2\rangle$, $\rho_2 = |\Phi_3\rangle$, $p_1 = p_2 = 1/2$, shows that C_{Tr} is not a strong monotone.

What about E_{Tr}?

$$E_{\mathrm{Tr}}(\rho) := \min_{\sigma \in \{\text{Separable states}\}} \|\rho - \sigma\|_1$$

- Like C_{Tr}, an entanglement monotone.
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E_{Tr} is not a strong monotone [Qiao *et al.*, arXiv: 1710.04447 (2017)]!

Facts about C_{ℓ_1}

- Captures the simple intuitive idea that superposition corresponds to off-diagonals.
- Easily computable even in the time-dependent case where the evolved density matrix can not be diagonalized, so eigenvalues and C_r become non-computable.
- Physical: Success probability of unambiguous state discrimination in interference experiments, 'which-path information' about a particle inside a multipath interferometer [Bagan *et al.* PRL (2016)].
- No conspicuous role in entanglement theory

C_{ℓ_1} for pure states

- Equals to Negativity: For pure states, C_{ℓ_1} is negativity of the *corresponding* bipartite state,

$$C_{\ell_1}\left(|\psi\rangle := \sum\sqrt{\lambda_i}|i\rangle\right) = \left(\sum\sqrt{\lambda_i}\right)^2 - 1 = 2\mathcal{N}\left(|\phi\rangle := \sum\sqrt{\lambda_i}|i\,i\rangle\right)$$

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- Upper bounds C_d:

$$C_d\left(|\psi\rangle\right) = C_r\left(|\psi\rangle\right) \leq C_{\ell_1}\left(|\psi\rangle\right), \quad \forall |\psi\rangle.$$

Equality holds iff diag$[|\psi\rangle\langle\psi|]$ is (up to permutation) either $\{1, 0, \cdots, 0\}$, or, $\{1/2, 1/2, 0, \ldots, 0\}$. That is, either $|\psi\rangle \in \mathcal{I}$, or $|\psi\rangle \equiv |\Phi_2\rangle \oplus 0$.

Using the recursive property [Lin, IEEE Trans. Inf. Theory, 1991] of entropy function $H(\lambda)$, we have

$$C_{\ell_1}\left(|\psi\rangle = \sum_{i=1}^{d}\sqrt{\lambda_i}|i\rangle\right) - C_r(|\psi\rangle)$$

$$= 2\sum_{i=1}^{d-1}\sqrt{\lambda_i}\sum_{j=i+1}^{d}\sqrt{\lambda_j} - H(\lambda)$$

$$\geq 2\sum_{i=1}^{d-1}\sqrt{\lambda_i}\sqrt{\sum_{j=i+1}^{d}\lambda_j} - H(\lambda)$$

$$= \sum_{i=1}^{d-1}\left[\left(\sum_{k=i}^{d}\lambda_k\right)\underbrace{\left(2\sqrt{\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\left(1-\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\right)} - H_2\left(\frac{\lambda_i}{\sum_{k=i}^{d}\lambda_k}\right)\right)}_{\geq 0,\ \text{as } H_2(x)\leq 2\sqrt{x(1-x)}}\right]. \quad \square$$

- The function $C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle)$ is Schur-concave in diag$(|\psi\rangle\langle|\psi|)$.

$$\boxed{0 \leq C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle) \leq d - 1 - \log_2 d, \quad d = \mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)].}$$

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- All $|\psi\rangle$ with $C_{\ell_1}(|\psi\rangle) = b$, has $C_r(|\psi\rangle)$ bounded by

$$\boxed{\frac{\sqrt{2}b^2}{d(d-1)} \le C_r \le \log_2(1+b), \quad d = \text{rank}[\text{diag}(|\psi\rangle\langle\psi|)]}$$

Note: Lower bound $\to 0$ as $d \to \infty$, but the upper bound does not depend on d.

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What is optimum C_r given only the knowledge of C_{ℓ_1}?

Optimum C_r from the knowledge of $C_{\ell_1}(|\psi\rangle)$

$$H_2(\alpha)+(1-\alpha)\log_2(d-1) \le C_r(|\psi\rangle) \le H_2(\beta)+(1-\beta)\log_2(n-1),$$

where $\alpha = \dfrac{2+(d-2)(d-b)+2\sqrt{(b+1)(d-1)(d-1-b)}}{d^2}$,

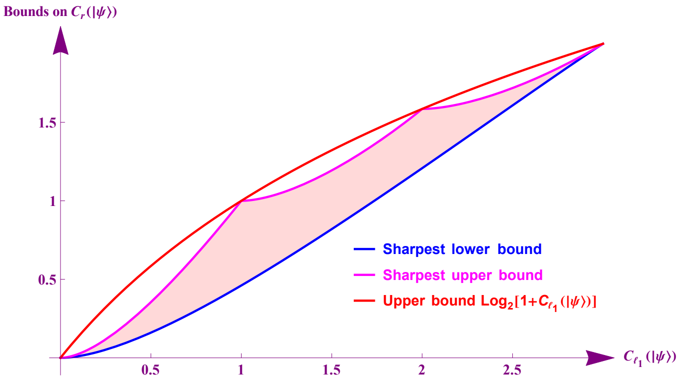
$$\beta = \frac{2+(n-2)(n-b)-2\sqrt{(b+1)(n-1)(n-1-b)}}{n^2},$$

$$d = \text{rank}[\text{diag}(|\psi\rangle\langle\psi|)],$$

$$n = \begin{cases} b+1 & \text{if } b \text{ is integer,} \\ [b]+2 & \text{otherwise,} \end{cases}$$

with $[x]$ denoting the integer part of x.

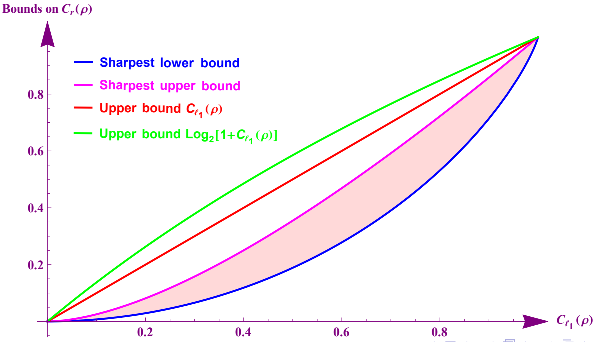
- Each of the bounds is satisfied by a unique state, up to permutation the diagonal elements of the state with minimum C_r are given by $\{\alpha, (1-\alpha)/(d-1), (1-\alpha)/(d-1), \cdots, (1-\alpha)/(d-1)\}$ and that with maximum C_r are $\{\beta, (1-\beta)/(n-1), (1-\beta)/(n-1), \cdots, (1-\beta)/(n-1)\}$.
- As $d \to \infty$, $\alpha \to 1$, so C_r could be arbitrarily small for large enough d, but can not be increased beyond that sharpest upper bound.



Optimum C_r from the knowledge of $C_{\ell_1}(\rho)$ for qubit ρ

- All qubit states ρ with a given coherence $C_{\ell_1}(\rho) = 2b$ satisfy

$$1 - H_2\left(\frac{1-2b}{2}\right) \leq C_r(\rho) \leq H_2\left(\frac{1-\sqrt{1-4b^2}}{2}\right) \leq C_{\ell_1}(\rho).$$



Optimum C_r from the knowledge of $C_{\ell_1}(\rho)$ for general ρ?

All we have is:

$$C_r(\rho) \leq \log_2\left[1 + C_R(\rho)\right] \leq \log_2\left[1 + C_{\ell_1}(\rho)\right] \leq \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \geq 1 \\ \frac{1}{\ln 2} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) < 1 \end{cases},$$

where C_R is the robustness, a strong monotone, defined by

$$C_R(\rho) := \min_{\sigma}\left\{ s \geq 0 \,\middle|\, \frac{\rho + s\sigma}{1+s} \in \mathscr{I} \right\} = \min_{\tau \in \mathscr{I}}\left\{ s \geq 0 \,\middle|\, \rho \leq (1+s)\tau \right\}.$$

Conclusion and Outlook

- Both ℓ_p- and Schatten-p-norm based functions are not strong monotone of coherence for any $p > 1$. Neither is a monotone in the first place. C_{Tr} is monotone, but not strong monotone like C_{ℓ_1}.

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- E_{Tr} is also monotone but not strong monotone. In [Eisert *et al.* JPA 2003] it is claimed that if we restrict the minimization over separable states having the same reductions as ρ, then the modified E_{Tr} is a strong monotone. What is an easy modification for C_{Tr} to make it a strong monotone?

Conclusion and Outlook

- Both ℓ_p- and Schatten-p-norm based functions are not strong monotone of coherence for any $p>1$. Neither is a monotone in the first place. C_{Tr} is monotone, but not strong monotone like C_{ℓ_1}.
- E_{Tr} is also monotone but not strong monotone. In [Eisert *et al.* JPA 2003] it is claimed that if we restrict the minimization over separable states having the same reductions as ρ, then the modified E_{Tr} is a strong monotone. What is an easy modification for C_{Tr} to make it a strong monotone?
- It seems that like convexity (as logarithmic version of any strong monotone is non-convex), strong monotonicity should be only a desirable feature for monotone, not a primary requirement.

- Operational (information theoretic) interpretations of C_{ℓ_1}:

$$C_d\left(|\psi\rangle\right) = C_r\left(|\psi\rangle\right) \leq C_{l_1}\left(|\psi\rangle\right), \forall\, |\psi\rangle$$

$$C_d(\rho) = C_r(\rho) \leq \log_2\left[1 + C_{\ell_1}(\rho)\right] \leq \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \geq 1 \\ \frac{1}{\ln 2} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) < 1 \end{cases}.$$

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Question: Can we drop the factor $1/\ln 2 \approx 1.4427$?

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Question: Can we drop the factor $1/\ln 2 \approx 1.4427$?

- If so, it would be sharpest as for any $0 < b < 1$ and $d \geq 3$, the d-dimensional state

$$\rho = \begin{pmatrix} b/2 & b/2 \\ b/2 & b/2 \end{pmatrix} \oplus (1-b)\delta, \quad \delta \in \mathscr{I}$$

has $C_r(\rho) = C_{\ell_1}(\rho) = b$

Thank You!