Intro to Coherence measures Trace-distance measure of coherence Relevance of C_{ℓ} Conclusioh

Title Plan

Trace-distance measure of Coherence

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Joint works...

Collaborators: Maciej Lewenstein, Swapan Rana, and Andreas Winter

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APS	Journals 👻	Help/Feedback	Journal, vol, page, DOI, etc.
		PHYSICAL REVIEW A covering atomic, molecular, and optical physics and quantum information	
		Trace-distance measure of coherence Swapan Rana, Preet Persohar, and Maciej, Levenstein Phys. Rev. A 93 , 012110 – Published 12 January 2016	y [] < More
APS	Journals 👻	Help/Feedback:	Journal, vol, page, DOI, etc.
		PHYSICAL REVIEW A covering atomic, molecular, and optical physics and quantum information	
		Logarithmic coherence: Operational interpretation of ℓ_1 -norm	
		Coherence Swapan Rana, Preeti Parashar, Andreas Winter, and Maciej Lewenstein	2
		Swapan halls, Free Fatasia, Autores Willer, and madej Levelisten Phys. Rev. A 96, 052336 – Published 27 November 2017	Y 🖪 < More

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Outline

Introduction to coherence measures

- Model: Incoherent Operations (IO)
- Coherence measures: basic requirements

Distance-based measures

- ℓ_p and Shatten-*p*-norm based measures?
- Neither is strong monotone for p > 1

► Trace distance measure of coherence

- Strong monotone for Qubits and X-states
- But not for all states
- What about entanglement?

• Operational interpretations of C_{ℓ_1}

• Equals to Negativity of the MC state

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• Optimum C_r for a given C_{ℓ_1} ?

Conclusion

Incoherent operations Monotones of coherence Distance-based measures: General rule

Coherence Theory

Resource Theory of Coherence [Baumgratz et al., PRL (2014)]

Free states: Diagonal in chosen basis

- 'Preferred', or 'Chosen' basis $\{|i\rangle\}$ of the corresponding Hilbert space $\mathcal{H} = \mathbb{C}^d$.
- Set of 'incoherent', or 'free' states:

$$\mathscr{I} = \left\{ \delta \left| \delta = \sum_{i} p_{i} |i\rangle \langle i| \right\}.$$

- Free Operations: Incoherent Operations (IO)
 - An operator K is 'incoherent' if $K\delta K^{\dagger} \in \mathcal{I}$, $\forall \delta \in \mathcal{I}$.
 - \implies K can have at most one non-zero entry in any column.
 - A CPTP map Λ is 'free' or 'incoherent' if \exists incoherent Kraus operators $\{K_n\}$ for Λ .

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Incoherent operations Monotones of coherence Distance-based measures: General rule

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What makes a coherence measure?

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What makes a coherence measure?

Coherence Monotones

Any function $C: \mathscr{B}(\mathcal{H}^d) \to \mathbb{R}^+$ is a coherence measure if it satisfies

1. Faithfulness: $C(\delta) = 0 \Leftrightarrow \delta \in \mathscr{I}$.

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- 3a. Monotonicity: $C(\Lambda_{\text{ICPTP}}[\rho]) \leq C(\rho), \quad \forall \rho \text{ and incoherent } \Lambda.$

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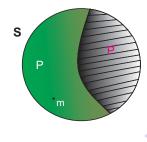
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- 3a. Monotonicity: $C(\Lambda_{\text{ICPTP}}[\rho]) \leq C(\rho), \quad \forall \rho \text{ and incoherent } \Lambda.$
- **3b.** Strong Monotonicity: $\sum p_n C(\rho_n) \le C(\rho)$, $\rho_n = K_n \rho K_n^{\dagger} / p_n$, $p_n = \text{Tr}[K_n \rho K_n^{\dagger}]$, for all incoherent {K_n}. Allows selective measurements or stochastic operations. Harder to verify: 3b. + 2. ⇒ 3a.

Incoherent operations Monotones of coherence Distance-based measures: General rule

Distance-based measures

How to quantify (measure)?

Let $m \in S$ and P be a *property* regarding the members of S. The usual way to quantify how much of the property P is contained in m is to determine the distance of m from the set of all $x \in S$ which does not have the property P.



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Incoherent operations Monotones of coherence Distance-based measures: General rule

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Entanglement from contractive metric [Vedral and Plenio, PRA 1998]

If a distance function ${\mathcal D}$ satisfies

- i. Positivity: $\mathcal{D}(\rho, \sigma) \ge 0 \quad \forall \ \rho, \sigma$, with equality iff $\rho = \sigma$,
- ii. Contractivity: $\mathcal{D}(\Lambda[\rho], \Lambda[\sigma]) \leq \mathcal{D}(\rho, \sigma)$ for all CPTP map Λ ,

then an entanglement measure can be defined through this distance as

$$E(\rho) = \inf_{\sigma \in \{\text{Separable states}\}} \mathscr{D}(\rho, \sigma).$$

✓ REE :

$$\mathscr{D}(x, y) = S(x \| y) := \begin{cases} \operatorname{Tr}(x \log x - x \log y), & \text{if support } x \subseteq \text{ support } y \\ +\infty, & \text{otherwise} \end{cases}$$

- ✓ Bures metric: $D(x, y) = 2 2\sqrt{F(x, y)}$, where $F(x, y) := [\text{Tr}{\sqrt{x}y\sqrt{x}}^{1/2}]^2$
- ✓ Trace distance: $D(x, y) = ||x y||_1 := \text{Tr} |x y|$

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- ✓ Trace distance: $D(x, y) = ||x y||_1 := \text{Tr} |x y|$
- ★ Hilbert-Schmidt distance:
 D(x, y) = ||x y||₂ is not-contractive [Ozawa, PLA 2000]

 $C_{\ell p}, C_p$ Nether is a (strong) monotone for p > 1 C_{Tr} is not a strong monotone Entanglement...?

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Measures from ℓ_p - and Schatten-p-norm (C_{ℓ_p}, C_p)

★ For $X \in \mathbb{C}^{m \times n}$, $p \in [1, \infty)$, the norms are defined by

$$\begin{split} \|X\|_{\ell_p} &:= \left(\sum_{i,j} |x_{ij}|^p\right)^{1/p}, \\ \|X\|_p &:= \left(\operatorname{Tr} |X|^p\right)^{1/p} = \left(\sum_i^r \sigma_i^p\right)^{1/p}, \sigma = \lambda\left(\sqrt{X^{\dagger}X}\right). \end{split}$$

The induced distance functions C_{lp} and C_p satisfies 1. and 2.
So we have to check only 3: the (strong) monotonicity.

 $\mathcal{L}_{f,p}$, \mathcal{C}_p Netther is a (strong) monotone for p > 1 \mathcal{L}_{Tr} is not a strong monotone Entanglement...?

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- * The induced distance functions C_{ℓ_n} and C_p satisfies 1. and 2.
- * So we have to check only 3: the (strong) monotonicity.
- Both are monotone on qubits. Is it so in higher dim?

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- * The induced distance functions C_{ℓ_p} and C_p satisfies 1. and 2.
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b For all *p* ∈ (1,∞) neither C_{ℓ_p} nor C_p is a monotone [PRA (2016)].

 $\begin{array}{l} \mathcal{C}_{\ell,n}, \mathcal{C}_p \\ \text{Neither is a (strong) monotone for } p > 1 \\ \mathcal{C}_{Tr} \text{ is not a strong monotone} \\ \text{Entanglement...?} \end{array}$

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None of C_{ℓ_p}, C_p is a monotone for p > 1

Counterexample from any $\rho \notin \mathscr{I}$

Let $K_i = |0\rangle\langle i - 1|$, i = 1, 2, ..., d and consider the incoherent channel $\Lambda = \{\tilde{K}_i\}$ with the Kraus operators $\tilde{K}_i = \mathbb{1} \otimes K_i$. Then we have

$$C_p \left(\Lambda[\rho \otimes \mathbb{1}/d] \right) = C_p \left(\rho \otimes |0\rangle \langle 0| \right)$$
$$= C_p \left(\rho \right)$$
$$> C_p \left(\rho \otimes \mathbb{1}/d \right).$$

The inequality follows from

$$C_p(\rho \otimes \mathbb{I}/d) \leq \|\rho \otimes \mathbb{I}/d - \delta^* \otimes \mathbb{I}/d\|_p = C_p(\rho) \|\mathbb{I}/d\|_p < C_p(\rho).$$

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As $C_{\ell_p}(\rho \otimes 1/d) = d^{1/p-1}C_{\ell_p}(\rho) < C_{\ell_p}(\rho)$, C_{ℓ_p} also violates monotonicity.

 $\begin{array}{l} C_{\ell,p}, C_p \\ \textbf{Neither is a (strong) monotone for } p > 1 \\ C_{Tr} \text{ is not a strong monotone} \\ \textbf{Entanglement}...? \end{array}$

Hence C_{ℓ_p}, C_p is not strong monotone for p > 1

Being convex, if it were strong monotone, it would be monotone!

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$$\rho = \frac{1}{4} \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \\ \bar{a} & 0 & 1 & 0 \\ 0 & \bar{b} & 0 & 1 \end{pmatrix}, K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$\rho_1 = \frac{1}{4p_1} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix} \oplus 0, \ \rho_2 = \frac{1}{4p_2} \begin{pmatrix} 1 & b \\ \bar{b} & 1 \end{pmatrix} \oplus 0, \ p_1 = p_2 = \frac{1}{2}.$$

The strong monotonicity (for both C_{ℓ_p} and C_p): $(|a|+|b|)^p \leq |a|^p + |b|^p$.

 $\begin{array}{l} C_{\ell,p}, C_p \\ \textbf{Neither is a (strong) monotone for } p > 1 \\ C_{Tr} \text{ is not a strong monotone} \\ \textbf{Entanglement}...? \end{array}$

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The strong monotonicity (for both C_{ℓ_p} and C_p): $(|a|+|b|)^p \leq |a|^p + |b|^p$.

$$(x+y)^{1+\epsilon} = (x+y)(x+y)^{\epsilon} = x(x+y)^{\epsilon} + y(x+y)^{\epsilon} > x \cdot x^{\epsilon} + y \cdot y^{\epsilon} = x^{1+\epsilon} + y^{1+\epsilon}$$

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The strong monotonicity (for both C_{ℓ_p} and C_p): $(|a|+|b|)^p \leq |a|^p + |b|^p$.

$$(x+y)^{1+\epsilon} = (x+y)(x+y)^{\epsilon} = x(x+y)^{\epsilon} + y(x+y)^{\epsilon} > x \cdot x^{\epsilon} + y \cdot y^{\epsilon} = x^{1+\epsilon} + y^{1+\epsilon}$$

Strong monotonicity does not hold for all $ab \neq 0$ and $p \in (1,\infty)$.

 $C_{\ell,p}, C_p$ Neither is a (strong) monotone for p > 1 C_{Tr} is not a strong monotone Entanglement...?

The only remaining case p = 1: C_{ℓ_1} and C_{Tr}

$$C_{\ell_1}(\rho) := \min_{\delta \in \mathscr{I}} \|\rho - \delta\|_{\ell_1}$$
$$= \sum_{i \neq j} |\rho_{ij}|$$

$$C_{\mathrm{Tr}}(\rho) := \min_{\delta \in \mathscr{I}} \|\rho - \delta\|$$
$$= ?$$

- C_{ℓ_1} is a (strong) monotone [Baumgratz *et al.*, PRL (2014)].
- C_{Tr} is a monotone, as $\|.\|_1$ is contractive under any CPTP map.

Semidefinite Program (SDP) for $C_{Tr}(\rho)$ [PRA (2016)]

 $\begin{array}{ll} \mbox{Minimize} & \mbox{Tr}(P+N) \\ \mbox{subject to} & \left\{ \begin{array}{ll} P-N &= \rho-\delta, \\ \mbox{Tr}\delta &= 1, \\ \delta & \in \mathscr{I} \\ P,N,\delta &\geq 0. \end{array} \right. \end{array}$

For pure states, a better SDP: [Chen et al., PRA (2016)].

 $\begin{array}{l} C_{\ell,p}, C_p \\ \text{Netther is a (strong) monotone for } p > 1 \\ C_{\text{Ir}} \text{ is not a strong monotone} \\ \text{Entanglement}...? \end{array}$

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C_{Tr} is monotone but not strong monotone

 C_{Tr} is strong monotone on qubits, X-states, and any direct sum (⊕) of those [PRA (2016)].

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Strong coherence monotone: any convex and monotone *C* is strong monotone iff $C(p_1\rho_1 \oplus p_2\rho_2) = p_1C(\rho_1) + p_2C(\rho_2)$.

For
$$|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum |i\rangle$$
, $C_{\text{Tr}}(|\Phi_d\rangle) = 2(1-1/d)$. Choosing $\rho_1 = |\Phi_2\rangle$,
 $\rho_2 = |\Phi_3\rangle$, $p_1 = p_2 = 1/2$, shows that C_{Tr} is not a strong monotone.

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What about $E_{\rm Tr}$?

$$E_{\mathrm{Tr}}(\rho) := \min_{\sigma \in \{ \mathrm{Separable \ states} \}} \|\rho - \sigma\|_1$$

- Like C_{Tr}, an entanglement monotone.
- But not computable like C_{Tr}.

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Same for pure states [Chen et al., PRA (2016)]:

$$C_{\rm Tr}\left(\sum \sqrt{\lambda_i} |i\rangle\right) = E_{\rm Tr}\left(\sum \sqrt{\lambda_i} |ii\rangle\right)$$

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 E_{Tr} is not a strong monotone [Qiao *et al.*, arXiv: 1710.04447 (2017)]!

The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

Facts about C_{ℓ_1}

- Captures the simple intuitive idea that superposition corresponds to off-diagonals.
- Easily computable even in the time-dependent case where the evolved density matrix can not be diagonalized, so eigenvalues and C_r become non-computable.
- Physical: Success probability of unambiguous state discrimination in interference experiments, 'which-path information' about a particle inside a multipath interferometer [Bagan *et al.* PRL (2016)].
- No conspicuous role in entanglement theory

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The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

C_{ℓ_1} for pure states

Equals to Negativity: For pure states, C_{ℓ_1} is negativity of the *corresponding* bipartite state,

$$C_{\ell_1}\left(|\psi\rangle := \sum \sqrt{\lambda_i}|i\rangle\right) = \left(\sum \sqrt{\lambda_i}\right)^2 - 1 = 2\mathcal{N}\left(|\phi\rangle := \sum \sqrt{\lambda_i}|ii\rangle\right)$$

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 \blacksquare Upper bounds C_d :

$$C_d(|\psi\rangle) = C_r(|\psi\rangle) \le C_{\ell_1}(|\psi\rangle), \quad \forall |\psi\rangle.$$

Equality holds iff diag[$|\psi\rangle\langle\psi|$] is (up to permutation) either $\{1,0,\cdots,0\}$, or, $\{1/2,1/2,0,\ldots,0\}$. That is, either $|\psi\rangle\in\mathscr{I}$, or $|\psi\rangle \equiv |\Phi_2\rangle\oplus 0$.

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 $\begin{array}{c} \mbox{Front matters} & \mbox{The simplest and intuitive} \\ \mbox{Intro to Coherence measure of coherence} \\ \mbox{Relevance of } C_{\ell_1} & \mbox{Qubits} \\ \mbox{Conclusion} & \mbox{General } \rho? \end{array}$

Using the recursive property [Lin, IEEE Trans. Inf. Theory, 1991] of entropy function $H(\lambda)$, we have

$$\begin{split} C_{\ell_1} \left(|\psi\rangle &= \sum_{i=1}^d \sqrt{\lambda_i} |i\rangle \right) - C_r(|\psi\rangle) \\ &= 2 \sum_{i=1}^{d-1} \sqrt{\lambda_i} \sum_{j=i+1}^d \sqrt{\lambda_j} - H(\lambda) \\ &\ge 2 \sum_{i=1}^{d-1} \sqrt{\lambda_i} \sqrt{\sum_{j=i+1}^d \lambda_j} - H(\lambda) \\ &= \sum_{i=1}^{d-1} \left[\left(\sum_{k=i}^d \lambda_k \right) \underbrace{\left(2 \sqrt{\frac{\lambda_i}{\sum_{k=i}^d \lambda_k} \left(1 - \frac{\lambda_i}{\sum_{k=i}^d \lambda_k} \right) - H_2 \left(\frac{\lambda_i}{\sum_{k=i}^d \lambda_k} \right) \right]}_{\ge 0, \text{ as } H_2(x) \le 2\sqrt{x(1-x)}} \right]. \quad \Box$$

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Front matters The simpl Intro to Coherence measures Pure state Trace-distance measure of coherence Optimum Relevance of C_ℓ, Conclusio General ρ

The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

b The function $C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle)$ is Schur-concave in diag $(|\psi\rangle\langle|\psi|)$.

 $0 \leq C_{\ell_1}(|\psi\rangle) - C_r(|\psi\rangle) \leq d-1 - \log_2 d, \quad d = \mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)].$

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 \Rightarrow All $|\psi\rangle$ with $C_{\ell_1}(|\psi\rangle) = b$, has $C_r(|\psi\rangle)$ bounded by

$$\frac{\sqrt{2}b^2}{d(d-1)} \leq C_r \leq \log_2(1+b), \quad d = \mathrm{rank}[\mathrm{diag}(|\psi\rangle\langle\psi|)]$$

Note: Lower bound $\rightarrow 0$ as $d \rightarrow \infty$, but the upper bound does not depend on d.

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What is optimum C_r given only the knowledge of C_{ℓ_1} ?

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The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

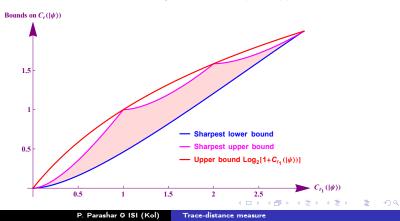
Optimum C_r from the knowledge of $C_{\ell_1}(|\psi\rangle)$

$$\begin{split} H_{2}(\alpha) + (1-\alpha)\log_{2}(d-1) &\leq C_{r}(|\psi\rangle) \leq H_{2}(\beta) + (1-\beta)\log_{2}(n-1), \\ \text{where } \alpha &= \frac{2 + (d-2)(d-b) + 2\sqrt{(b+1)(d-1)(d-1-b)}}{d^{2}}, \\ \beta &= \frac{2 + (n-2)(n-b) - 2\sqrt{(b+1)(n-1)(n-1-b)}}{n^{2}}, \\ d &= \operatorname{rank}[\operatorname{diag}(|\psi\rangle\langle\psi|)], \\ n &= \begin{cases} b+1 & \text{if } b \text{ is integer}, \\ [b]+2 & \text{otherwise,} \end{cases} \\ \text{with } [x] \text{ denoting the integer part of } x. \end{split}$$

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- Each of the bounds is satisfied by a unique state, up to permutation the diagonal elements of the state with minimum C_r are given by $\{\alpha, (1-\alpha)/(d-1), (1-\alpha)/(d-1), \cdots, (1-\alpha)/(d-1)\}$ and that with maximum C_r are $\{\beta, (1-\beta)/(n-1), (1-\beta)/(n-1), \cdots, (1-\beta)/(n-1)\}$.
- As d→∞, α→1, so C_r could be arbitrarily small for large enough d, but can not be increased beyond that sharpest upper bound.



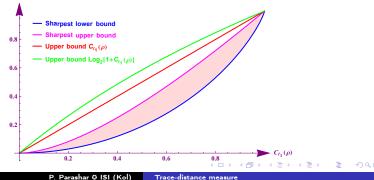
The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

Optimum C_r from the knowledge of $C_{\ell_1}(\rho)$ for qubit ρ

Solution All qubit states ρ with a given coherence $C_{\ell_1}(\rho) = 2b$ satisfy

$$1 - H_2\left(\frac{1 - 2b}{2}\right) \le C_r(\rho) \le H_2\left(\frac{1 - \sqrt{1 - 4b^2}}{2}\right) \le C_{\ell_1}(\rho).$$

Bounds on $C_r(\rho)$



Front matters Intro to Coherence measures Trace-distance measure of coherence Relevance of \mathcal{L}_{0} Conclusion The simplest and intuitive Pure states Optimum C_r from C_{ℓ_1} Qubits General ρ ?

Optimum C_r from the knowledge of $C_{\ell_1}(\rho)$ for general ρ ?

All we have is:

$$C_r(\rho) \le \log_2 \left[1 + C_R(\rho) \right] \le \log_2 \left[1 + C_{\ell_1}(\rho) \right] \le \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \ge 1 \\ \frac{1}{\ln 2} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) < 1 \end{cases},$$

where C_R is the robustness, a strong monotone, defined by

$$C_R(\rho) := \min_{\sigma} \left\{ s \ge 0 \mid \frac{\rho + s\sigma}{1 + s} \in \mathscr{I} \right\} = \min_{\tau \in \mathscr{I}} \left\{ s \ge 0 \mid \rho \le (1 + s)\tau \right\}.$$

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Conclusion Thanks

Conclusion and Outlook

 Both ℓ_p- and Schatten-p-norm based functions are not strong monotone of coherence for any p > 1. Neither is a monotone in the first place. C_{Tr} is monotone, but not strong monotone like C_{ℓ1}.

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- $E_{\rm Tr}$ is also monotone but not strong monotone. In [Eisert *et al.* JPA 2003] it is claimed that if we restrict the minimization over separable states having the same reductions as ρ , then the modified $E_{\rm Tr}$ is a strong monotone. What is an easy modification for $C_{\rm Tr}$ to make it a strong monotone?

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- $E_{\rm Tr}$ is also monotone but not strong monotone. In [Eisert *et al.* JPA 2003] it is claimed that if we restrict the minimization over separable states having the same reductions as ρ , then the modified $E_{\rm Tr}$ is a strong monotone. What is an easy modification for $C_{\rm Tr}$ to make it a strong monotone?
- It seems that like convexity (as logarithmic version of any strong monotone is non-convex), strong monotonicity should be only a desirable feature for monotone, not a primary requirement.

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• Operational (information theoretic) interpretations of C_{ℓ_1} :

$$\begin{split} C_d\left(\left|\psi\right\rangle\right) &= C_r\left(\left|\psi\right\rangle\right) \leq C_{l_1}\left(\left|\psi\right\rangle\right), \,\forall \left|\psi\right\rangle \\ C_d(\rho) &= C_r(\rho) \leq \log_2\left[1 + C_{\ell_1}\left(\rho\right)\right] \leq \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \geq 1\\ \frac{1}{\ln 2}C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) < 1 \end{cases} \end{split}$$

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Question: Can we drop the factor $1/\ln 2 \approx 1.4427$?

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Question: Can we drop the factor $1/\ln 2 \approx 1.4427$?

• If so, it would be sharpest as for any 0 < b < 1 and $d \ge 3$, the d-dimensional state

$$\rho = \begin{pmatrix} b/2 & b/2 \\ b/2 & b/2 \end{pmatrix} \oplus (1-b)\delta, \quad \delta \in \mathcal{I}$$

has $C_r(\rho) = C_{\ell_1}(\rho) = b$

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Conclusion Thanks

Thank You!

P. Parashar Ø ISI (Kol) Trace-distance measure

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